Workshop on group schemes and p-divisible groups: Homework 3.

1. Let K be the fraction field of a complete discrete valuation ring with mixed characteristic $(0,p)$ and perfect residue field. Let $\overline{K}/K$ be an algebraic closure and $\mathbf{C}_K$ its completion. Let $G_K = \mathrm{Gal}(\overline{K}/K)$, so G_K acts on $\mathbf{C}_K$ be isometries in the evident manner.

(i) Prove that if V is a finite-dimensional $\mathbf{C}_K$-vector space endowed with a continuous semi-linear G_K-action, then the natural $\mathbf{C}_K$-linear G_K-equivariant map $\oplus_{i\in\mathbf{Z}}(\mathbf{C}_K \otimes_K V(i)^{G_K}) \to V$ is injective, where $V(i) = V \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1)^{\otimes i}$. (Hint: consider a nonzero element in the kernel that is a sum of a minimal number of elementary tensors.)

(ii) Prove that if $\chi : G_K \to K^\times$ has finite order then the semi-linear Galois module $\mathbf{C}_K(\chi)$ given by $\mathbf{C}_K$ endowed with the action $[g](c) = \chi(g)g(c)$ has a nonzero G_K-invariant vector, so $\mathbf{C}_K(\chi) \simeq \mathbf{C}_K$ as semi-linear Galois modules. What happens if $\mathbf{C}_K$ is replaced with $\overline{K}$ (and $\chi \neq 1$)?

(iii) Let E be an elliptic curve over K with multiplicative reduction, so as a $\mathbf{Q}_p[G_K]$-module $\mathrm{V}_p(E)$ is an extension of $\mathbf{Q}_p(\chi)$ by $\mathbf{Q}_p(\chi\varepsilon_p)$ where $\chi^2 = 1$ and ε_p is the p-adic cyclotomic character. Prove that this extension structure becomes semi-linearly split after extending scalars to $\mathbf{C}_K$. (Using the Serre–Tate equivalence between the deformation theory of an abelian variety and its p-divisible group in residue characteristic p, one can construct many examples of such E for which $\overline{K} \otimes_{\mathbf{Q}_p} \mathrm{V}_p(E)$ is a non-split semilinear extension.)

2. Let $F/\mathbf{Q}_p$ have degree d, and let K be the fraction field of a complete discrete valuation ring R with mixed characteristic $(0,p)$. Let $\overline{K}/K$ be an algebraic closure.

(i) Let G be a p-divisible group over R with height d. Assume there is given an action by $\mathscr{O}_F$ on G. Prove that $\mathrm{T}_p(G) = \varprojlim G[p^n](\overline{K})$ is a free $\mathscr{O}_F$-module of rank 1. Deduce also that $\mathscr{O}_F$ acts on the étale and connected parts of G, and conclude that G is either étale or connected.

(ii) Let A be an abelian scheme over R with relative dimension g. Assume that there is given an $\mathscr{O}_F$-action on A, and that $d = 2g$. Using that $\det \mathrm{T}_p(A) = \varepsilon_p^g$ (which comes out of the study of the étale cohomology of abelian varieties), deduce that $A[p^\infty]$ must be connected and that the action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{T}_p(A)$ is given by a continuous $\mathscr{O}_F^\times$-valued character. (In particular, the splitting field for $A_K[p^\infty]$ is an *abelian* extension of K.)

3. Let S be a scheme.

(i) If $0 \to G' \to G \to G'' \to 0$ and $0 \to H' \to H \to H'' \to 0$ are diagrams of finite locally free commutative S-groups, prove that $0 \to H' \times G' \to H \times G \to H'' \times G'' \to 0$ is short exact if and only if the two given diagrams are short exact.

(ii) Use the viewpoint of *fppf* abelian sheaves to prove that if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 0 \\ & & \simeq\downarrow & & \downarrow & & \downarrow\simeq & & \\ 0 & \longrightarrow & H' & \longrightarrow & H & \longrightarrow & H'' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of short exact sequences and the outer maps are isomorphisms then so is the middle map.

(iii) Explain step by step how to construct the inverse map $H \to G$ in (ii) via descent theory. Then give a third proof by passing to geometric fibers over S.

4. Let k be a field and let X be a proper k-scheme satisfying $\mathscr{O}_X(X) = k$, and assume $X(k)$ is non-empty, so upon choosing $e \in X(k)$ we get a representable Picard functor $\mathrm{Pic}_{X/k,e}$ that is locally of finite type over k.

(i) If Z is any k-scheme locally of finite type and $z \in Z(k)$, identify the tangent space $\mathrm{T}_z(Z)$ with the fiber over z for the map of sets $Z(k[\varepsilon]) \to Z(k)$. Define operations on the k-algebra $k[\varepsilon]$ that thereby enhance this set-theoretic bijection to a k-linear isomorphism. (Hint: For the fiber-product ring $k[\varepsilon] \times_k k[\varepsilon']$, note that the natural map $Z(k[\varepsilon] \times_k k[\varepsilon']) \to Z(k[\varepsilon]) \times_{Z(k)} Z(k[\varepsilon'])$ is bijective.)

(*ii*) Prove that the bijection $\mathrm{T}_0(\mathrm{Pic}_{X/k,e}) \simeq \mathrm{H}^1(X, \mathscr{O}_X)$ from lecture is a *k-linear* isomorphism.

5. (*i*) Use the fibral flatness criterion to prove that if $A \to S$ is an abelian scheme with relative dimension $g > 0$ and if $N \in \mathbf{Z} - \{0\}$ then $N : A \to A$ is a finite locally free map with degree $|N|^{2g}$. Deduce that $A[N]$ is a finite locally free commutative N-torsion S-group of order $|N|^{2g}$.

(*ii*) Prove that for any prime p, the directed system $A[p^\infty] = \{A[p^n]\}$ is a p-divisible group of height $2g$ over S.

(*iii*) If $S = \mathrm{Spec}(R)$ for a complete local noetherian ring R, by HW2, Exercise 9 we get a commutative formal Lie group $\mathscr{O}^\wedge_{A,0}$ by completing at the identity on the closed fiber. Prove that this "is" the identity component of $A[p^\infty]$ in the sense of the Serre–Tate equivalence, and so this p-divisible group has relative dimension g over R.

6. This exercise works out descent theory for quasi-coherent sheaves. Let $f : S' \to S$ be a morphism of schemes and let $S'' = S' \times_S S'$ be equipped with the projections $p_1, p_2 : S'' \rightrightarrows S'$. For an $\mathscr{O}_{S'}$-module $\mathscr{F}'$, a *descent datum* on $\mathscr{F}'$ is an isomorphism $\varphi : p_2^*(\mathscr{F}') \simeq p_1^*(\mathscr{F}')$ that satisfies the *cocycle condition* over the triple product S''' (equipped with projections $q_{ij} : S''' \to S''$ for $1 \le i < j \le 3$): $q_{13}^*(\varphi) = q_{12}^*(\varphi) \circ q_{23}^*(\varphi)$. A *morphism* of pairs $(\mathscr{F}_1', \varphi_1) \to (\mathscr{F}_2', \varphi_2)$ is an $\mathscr{O}_{S'}$-linear map $\mathscr{F}_1' \to \mathscr{F}_2'$ whose pullbacks under the p_i's are compatible with the descent data in the evident manner (a certain square commutes).

(*i*) In the case $S' = \coprod S_i$ for an open covering $\{S_i\}$ of S, explain why a descent datum on an $\mathscr{O}_{S'}$-module is the same as classical gluing data for $\mathscr{O}_{S_i}$-modules on the S_i's.

(*ii*) If $\mathscr{F}$ is an $\mathscr{O}_S$-module, explain how to give $\mathscr{F}' = f^*(\mathscr{F})$ a canonical descent datum in a manner that is functorial in $\mathscr{F}$.

(*iv*) If $S' \to S$ is faithfully flat and quasi-compact, prove that the functor constructed in (*ii*) is fully faithful when working with quasi-coherent sheaves. (It is an important result of Grothendieck that this is actually an equivalence of categories in the quasi-coherent setting; reduce this problem to the affine case. Can you solve the affine case?)

7. This exercise develops the properties of a very useful notion called the *Serre tensor construction*. Let S be a scheme and let A be a commutative ring. (It is of interest to allow some non-commutative rings, but we suppress such generality here.) Let X be an A-module scheme over S and M a projective A-module of finite rank.

(*i*) Using a finite presentation for the dual linear module, prove that the functor $T \rightsquigarrow M \otimes_A X(T)$ is represented by an A-module scheme over S; it is denoted $M \otimes_A X$. Check that $M \otimes_A (\cdot)$ carries closed immersions to closed immersions, surjections to surjections, and commutes with formation of fiber products in X. How does it behave with respect to direct sums in M?

(*ii*) Using Serre's trick from HW2, Exercise 2, prove that if X is S-flat then so is $M \otimes_A X$. (Hint: Express M as a direct summand of a free module.) Is this flatness obvious via the *construction* of $M \otimes_A X$? Also check preservation of separatedness and properness via the valuative criterion, and the property of being locally of finite type (resp. locally of finite presentation).

(*iii*) By expressing M as a direct summand of a finite free A-module, show that if X is locally of finite type over S with geometrically connected fibers, then the same holds for $M \otimes_A X$. Is this obvious via the *construction* of $M \otimes_A X$?

(*iv*) Using the functorial description of tangent spaces and the functorial criterion for smoothness, prove that $M \otimes_A \mathrm{T}_0(X) \simeq \mathrm{T}_0(M \otimes_A X)$ and that if X is smooth over an algebraically closed field then so is $M \otimes_A X$ and moreover in such cases we have $\dim(M \otimes_A X) = r \dim X$ if M has constant rank r over A.

(v) Explain how the Serre construction behaves with respect to short exact sequences of finite locally free commutative group schemes, Cartier duality, Dieudonné modules, abelian schemes, Tate modules of abelian varieties, and complex-analytic uniformizations. All isomorphisms should be constructed in a canonical manner.

(vi) If X is an abelian variety over a field k and A is its own centralizer in $\mathrm{End}_k(X)$, prove that the natural map $M \to \mathrm{Hom}_{k,A}(X, M \otimes_A X)$ to the space of *A-linear* k-homorphisms is an isomorphism. In particular, deduce that if M and M' are invertible A-modules, then $M \otimes_A X$ and $M' \otimes_A X$ are *A-linearly* k-isomorphic if and only if M and M' are A-linearly isomorphic.

8. Let X and Y be abelian varieties over a field k and let K/k be an extension field. Assume either that k is separably closed or that K/k is purely inseparable. (Important examples are $k = \overline{\mathbf{Q}}$ and $K = \mathbf{C}$, or k imperfect and K its perfect closure.)

(i) Prove that $\mathrm{Spec}(K \otimes_k K)$ is connected.

(ii) Fix a prime $\ell \neq \mathrm{char}(k)$ and let $f : X_K \to Y_K$ be a map of abelian varieties. Use (i) to prove that if k is separably closed then $p_1^*(f), p_2^*(f) : X_{K\otimes_k K} \rightrightarrows Y_{K\otimes_k K}$ coincide on the constant (!) ℓ^n-torsion subgroup schemes for all $n \geq 1$ by comparing them over the diagonal point. Conclude that these maps coincide on all fibers. Do the same without restriction on k for K/k purely inseparable by using Galois theory to descend from the setup for $(k_{\mathrm{s}} \otimes_k K)/k_{\mathrm{s}}$.

(iii) By reducing to K/k finitely generated (so $K \otimes_k K$ is noetherian), deduce $p_1^*(f) = p_2^*(f)$, so by *fpqc* descent f uniquely descends to a k-map $X \to Y$ that is also a k-group map! In particular, for an abelian variety over K there is at most one descent to an abelian variety over k and such a descent is *functorial* if it exists. Thus, it is unambiguous to speak of "the" descent to k for an abelian variety over K (if one exists); what if $k = \mathbf{Q}$?

9. Let A be an abelian variety over an algebraically closed field k and let K/k be an extension.

(i) If $\dim A = 1$, prove that any isogeny $A_K \to B$ over K is uniquely "defined" over k. (That is, any finite K-subgroup of A_K has the form G_K for a unique finite k-subgroup $G \subseteq A$.)

(ii) If $\mathrm{char}(K) = 0$, prove the same conclusion without restriction on $\dim A$.

(iii) Let $A = E \times E$ be a product of two supersingular elliptic curves. By studying order-p subgroups of $\alpha_p \times \alpha_p$, prove that if $K \neq k$ then A_K contains order-p subgroups not arising from A. Using Exercise 8, conclude that the quotient of A_K by such a subgroup *cannot* be defined over k as an abstract abelian variety (ignoring the isogeny with A_K).

10. This exercise proves the Poincaré reducibility theorem over an arbitrary ground field k, using Exercise 8 to handle imperect k. Let X be an abelian variety over k.

(i) If $\mathrm{End}_k^0(X)$ is not a division algebra, construct a nonzero map $f : X \to X$ that is not an isogeny. Show that the scheme-theoretic image $Y = f(X)$ is a nonzero proper abelian subvariety.

(ii) If $i : Y \hookrightarrow X$ is a nonzero proper abelian subvariety over k and if k is perfect, find an "isogeny complement" as follows. Pick an ample invertible sheaf $\mathscr{L}$ on X and let $Z = \ker(i^\vee \circ \phi_{\mathscr{L}})^0_{\mathrm{red}}$. Using perfectness, show that Z is an abelian subvariety of X (over k) with $\dim Z \geq \dim X - \dim Y$ (equality holds *a priori* but we do not need to check this) and explain why $i^\vee \circ \phi_{\mathscr{L}}|_Y = \phi_{\mathscr{L}|_Y}$ (an isogeny since $\mathscr{L}|_Y$ is ample). Deduce that $Y \cap Z$ is finite, so $Z \times Y \to X$ is an isogeny.

(iii) In the general case, if k_{p}/k is the perfect closure and Y is a nonzero proper abelian subvariety of X over k then find an "isogeny complement" as follows. Let $Z' \subseteq X_{k_{\mathrm{p}}}$ be an abelian subvariety over k_{p} such that $Y_{k_{\mathrm{p}}} \times Z' \to X_{k_{\mathrm{p}}}$ is an isogeny (use (ii) over k_{p}). Let $X_{k_{\mathrm{p}}} \to Y_{k_{\mathrm{p}}} \times Z'$ be an isogeny over k_{p}, and consider the composite k_{p}-map

$$X_{k_{\mathrm{p}}} \to Y_{k_{\mathrm{p}}} \times Z' \overset{\mathrm{pr}_2}{\to} Z' \hookrightarrow X_{k_{\mathrm{p}}}.$$

By Exercise 8, this composite map descends to $\mathrm{End}_k(X)$. Prove that its scheme-theoretic image is an abelian subvariety Z of X over k and that $Y \times Z \to X$ is an isogeny.

(*iv*) Prove the Poincaré reducibility theorem over k: every nonzero abelian variety over k is k-isogenous to a product $\prod X_i^{e_i}$ of pairwise non-isogenous abelian varieties X_i that are k-simple in the sense that X_i is nonzero and contains no proper nonzero abelian subvarieties over k, with all $e_i > 0$. Prove that the X_i's and e_i's are unique, and that $\mathrm{End}_k^0(X) \simeq \prod \mathrm{Mat}_{e_i}(D_i)$ where $D_i = \mathrm{End}_k^0(X_i)$ is a division algebra finite-dimensional over $\mathbf{Q}$.

(*v*) For K/k as in Exercise 8, use the trick in (*iii*) to prove that an abelian subvariety of X_K has the form Y_K for a unique abelian subvariety $Y \subseteq X$.